A study on matrices with hybrid number entries

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Abstract

In this paper, we firstly introduce the matrices with hybrid numbers entries. For the sake of compatibility with the literature, we will call this set of matrices as hybrid matrices. These matrices can be regarded as a generalization of complex matrices, dual matrices, and hyperbolic matrices. We then give basic properties of hybrid matrices by writing these matrices as a combination of real matrices. Finally, we examine the real matrix representation of hybrid matrices using the base elements.

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1 Introduction

Hybrid numbers are a generalization of complex, dual, and hyperbolic numbers. A hybrid number is denoted in the form $z = a_0 + a_1\mathbf{i} + a_2\boldsymbol{\varepsilon} + a_3\mathbf{h}$, where a_0, a_1, a_2, a_3 are real numbers [1]. It is obvious to see that a hybrid number is an any combination of complex, dual, and hyperbolic numbers, moreover the basis elements are $\{1, \mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}\}$. Set of hybrid numbers denoted by $\mathbb{K}$ and defined as

$$\mathbb{K} = \left\{ z = a_0 + a_1\mathbf{i} + a_2\boldsymbol{\varepsilon} + a_3\mathbf{h} : \ a_0, a_1, a_2, a_3 \in \mathbb{R}, \ \begin{array}{c} \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1 \\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \end{array} \right\}. \tag{1.1}$$

For every $z_1 = a_0 + a_1\mathbf{i} + a_2\boldsymbol{\varepsilon} + a_3\mathbf{h}$ and $z_2 = b_0 + b_1\mathbf{i} + b_2\boldsymbol{\varepsilon} + b_3\mathbf{h}$; equality, addition and multiplication are defined by

i) $z_1 = z_2 \iff a_0 = b_0, a_1 = b_1, a_2 = b_2, a_3 = b_3,$

ii) $z_1 + z_2 = (a_0 + b_0) + (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\boldsymbol{\varepsilon} + (a_3 + b_3)\mathbf{h},$

iii)
$$\begin{aligned} z_1 \cdot z_2 &= (a_0 + a_1\mathbf{i} + a_2\boldsymbol{\varepsilon} + a_3\mathbf{h}).(b_0 + b_1\mathbf{i} + b_2\boldsymbol{\varepsilon} + b_3\mathbf{h}) \\ &= (a_0b_0 - a_1b_1 + a_2b_1 - a_1b_2 + a_3b_3) \\ &\quad + (a_0b_2 + a_1b_0 + a_1b_3 - a_3b_1)\mathbf{i} \\ &\quad + (a_0b_2 + a_2b_0 - a_2b_3 + a_3b_2 + a_1b_3 - a_3b_1)\,\boldsymbol{\varepsilon} \\ &\quad + (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)\,\mathbf{h}\ . \end{aligned}$$

The addition operation is both commutative and associative. The null element is 0 and the inverse element of z is $-z$. As a result of these properties, $(\mathbb{K}, +)$ is an Abelian group. For the multiplication of hybrid units following table can be given:

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.	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$1+\mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Table 1. Multiplication table of hybrid units

One can easily understand that multiplication operation is not commutative but associative. The conjugate of a hybrid number z is denoted by $\overline{z}$ and defined by $\overline{z}=a_0-a_1\mathbf{i}-a_2\boldsymbol{\varepsilon}-a_3\mathbf{h}$. Moreover, $z\overline{z}=\overline{z}z$ satisfies and the character of a hybrid number z is denoted by $C(z)$ and defined as

$$C(z)=z\overline{z}=\overline{z}z=a_0^2+(a_1-a_2)^2-a_2^2-a_3^2 \tag{1.2}$$

where the result is a real number. Additionally, the modulus of a hybrid number $|z|$ is defined by the square root of $C(z)$. It can be seen that this modulus is a generalized modulus of the complex, dual, and hyperbolic number:

i) If $a_2=a_3=d=0$, then $|z|=\sqrt{a_0^2+a_1^2}$, (complex number)

ii) If $a_1=a_3=0$, then $|z|=\sqrt{a_0^2}$, (dual number)

iii) If $a_1=a_2=0$, then $|z|=\sqrt{|a_0^2-a_3^2|}$, (hyperbolic number)

More details on hybrid numbers can be found [1, 2, 3].

In the literature, there are different studies on the matrices of the different number systems. For example, dual matrices were defined by Veldkamp in [4], and their properties were studied in detail by Dağdeviren together with special dual matrices in [5]. Moreover, there are various articles in the literature on hyperbolic(perplex) numbers and their matrices [6, 7, 8, 9]. For further information about related studies, we may refer to the reader [5, 6, 7, 8, 9, 10]. We can summarize the literature again with the following table:

	Numbers	Matrices
Complex	$a+\boldsymbol{i}b\ ,\quad a,b\in\mathbb{R}$	$A+\boldsymbol{i}B\ ,\quad A,B\in\mathbb{R}_n^m$
Dual	$a+\boldsymbol{\varepsilon}b\ ,\quad a,b\in\mathbb{R}$	$A+\boldsymbol{\varepsilon}B\ ,\quad A,B\in\mathbb{R}_n^m$
Hyperbolic	$a+\boldsymbol{h}b\ ,\quad a,b\in\mathbb{R}$	$A+\boldsymbol{h}B\ ,\quad A,B\in\mathbb{R}_n^m$
Hybrid	$a+\boldsymbol{i}b+\boldsymbol{\varepsilon}c+\boldsymbol{h}d\ ,\quad a,b,c,d\in\mathbb{R}$	?

Table 2.

In this study, we will complete the gap which is shown with the question mark in Table 2. In this way, we will obtain the generalization form of all the other matrix sets(complex, dual, and hyperbolic matrices) which are given in the same Table. Moreover, we can present the outline of the study in the following paragraph.

In section 2, we will define hybrid matrices using the hybrid numbers. Additionally, in the same chapter, we will write hybrid matrices as a combination of real matrices and also give the definitions of trace, conjugate, and the transpose of hybrid matrices. In section 3, we will show the real matrix representations of hybrid matrices.

2 Hybrid matrices

In this section, we will define the set of hybrid matrices and examine their properties.

A hybrid matrix is a matrix with hybrid number entries. Any $m \times n$ hybrid matrix can be defined as:

$$\widehat{Z} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\ z_{21} & z_{22} & \cdots & z_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{m2} \end{bmatrix}, \quad z_{ij} \in \mathbb{K} \tag{2.1}$$

where $z_{ij} = a_{ij} + b_{ij}\mathbf{i} + c_{ij}\boldsymbol{\varepsilon} + d_{ij}\mathbf{h}$, $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$ and this matrix $\widehat{Z}$ can be written hybrid combination of four real matrix as follows:

$$\begin{aligned} \widehat{Z} = [Z_{ij}] &= [a_{ij} + \mathbf{i}b_{ij} + \boldsymbol{\varepsilon}c_{ij} + \mathbf{h}d_{ij}] \\ &= [a_{ij}] + \mathbf{i}\,[b_{ij}] + \boldsymbol{\varepsilon}\,[c_{ij}] + \mathbf{h}\,[d_{ij}] \\ &= A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D \end{aligned}$$

where $A, B, C, D \in \mathbb{R}_n^m$ matrices.

Example 2.1.

$$\widehat{Z} = \begin{bmatrix} 1 + 7\mathbf{i} + 3\mathbf{h} & 2 - \boldsymbol{\varepsilon} \\ -5 - 3\mathbf{h} & \mathbf{i} + \boldsymbol{\varepsilon} - \mathbf{i} \end{bmatrix}_{2\times 2}$$

matrix can be written as

$$\widehat{Z} = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} + \mathbf{i} \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} + \boldsymbol{\varepsilon} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \mathbf{h} \begin{bmatrix} 3 & 2 \\ -3 & -1 \end{bmatrix}.$$

Definition 2.2. For any hybrid matrix which all entries are zero called zero hybrid matrix and shown as $\widehat{0}$. Additionally, if the diagonal elements are $1 + 0\mathbf{i} + 0\boldsymbol{\varepsilon} + 0\mathbf{h}$ and the others are zero the regarding matrix called unit hybrid matrix and shown as $\widehat{I}$. Furthermore for every $\widehat{A} \in \mathbb{K}_n^n$, $\widehat{A} \cdot \widehat{I} = \widehat{I} \cdot \widehat{A} = \widehat{A}$.

Definition 2.3. Let $\widehat{A} = [A_{ij}]\,, \widehat{B} = [B_{ij}] \in \mathbb{K}_n^m$. Equality of these matrices is defined as:

$$\begin{aligned} \widehat{A} = \widehat{B} &\iff [A_{ij}] = [B_{ij}] \\ &\iff a_{ij} + b_{ij}\mathbf{i} + c_{ij}\boldsymbol{\varepsilon} + d_{ij}\mathbf{h} = a_{ij}^* + b_{ij}^*\mathbf{i} + c_{ij}^*\boldsymbol{\varepsilon} + d_{ij}^*\mathbf{h} \\ &\iff a_{ij} = a_{ij}^*, b_{ij} = b_{ij}^*, c_{ij} = c_{ij}^*, d_{ij} = d_{ij}^* \end{aligned}$$

where if $\widehat{A} = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D$ and $\widehat{B} = A^* + \mathbf{i}B^* + \boldsymbol{\varepsilon}C^* + \mathbf{h}D^*$ then

$$\begin{aligned} \widehat{A} = \widehat{B} &\iff A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D = A^* + \mathbf{i}B^* + \boldsymbol{\varepsilon}C^* + \mathbf{h}D^* \\ &\iff A = A^*, B = B^*, C = C^*, D = D^*. \end{aligned}$$

Definition 2.4. Let $\widehat{A} = [A_{ij}] = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D$ and $\widehat{B} = [B_{ij}] = A^* + \mathbf{i}B^* + \boldsymbol{\varepsilon}C^* + \mathbf{h}D^* \in \mathbb{K}_n^m$. Addition of $\widehat{A}$ and $\widehat{B}$ is defined as follows:

$$\begin{aligned} \widehat{A} + \widehat{B} &= [A_{ij}] + [B_{ij}] \\ &= (A + A^*) + \mathbf{i}(B + B^*) + \boldsymbol{\varepsilon}(C + C^*) + \mathbf{h}(D + D^*) \end{aligned}$$

Moreover, multiplication with a hybrid scalar $\lambda \in \mathbb{K}$ is

$$\lambda\widehat{A} = \lambda(A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D) = \lambda A + \lambda\mathbf{i}B + \lambda\boldsymbol{\varepsilon}C + \lambda\mathbf{h}D.$$

Theorem 2.5. Let $\widehat{A}, \widehat{B}, \widehat{C} \in \mathbb{K}_n^m$ and $z_1, z_2 \in \mathbb{K}$. The following equations hold:

i) $\widehat{A} + \widehat{B} = \widehat{B} + \widehat{A}$

ii) $\widehat{A} + (\widehat{B} + \widehat{C}) = (\widehat{A} + \widehat{B}) + \widehat{C}$

iii) $\widehat{A} + \widehat{0} = \widehat{A}$

iv) $\widehat{A} - \widehat{A} = \widehat{0}$

v) $z_1(\widehat{A} + \widehat{B}) = z_1\widehat{A} + z_1\widehat{B}$

vi) $(z_1 + z_2)\widehat{A} = z_1\widehat{A} + z_2\widehat{A}$

vii) $(z_1 z_2)\widehat{A} = z_1(z_2\widehat{A})$

Proof. The desired equations can be seen with the help of basic operations. Q.E.D.

Definition 2.6. Let $\widehat{A} \in \mathbb{K}_n^m$ and $\widehat{B} \in \mathbb{K}_p^n$. The multiplication of these hybrid matrices is defined as follows:

$$\begin{aligned}\widehat{A}\widehat{B} &= [a_{ij} + \mathbf{i}b_{ij} + \boldsymbol{\varepsilon}c_{ij} + \mathbf{h}d_{ij}]_{m\times n}\left[a^*_{jk} + \mathbf{i}b^*_{jk} + \boldsymbol{\varepsilon}c^*_{jk} + \mathbf{h}d^*_{jk}\right]_{n\times p} \\ &= \left[\sum_{j=1}^{n}(a_{ij} + \mathbf{i}b_{ij} + \boldsymbol{\varepsilon}c_{ij} + \mathbf{h}d_{ij})(a^*_{jk} + \mathbf{i}b^*_{jk} + \boldsymbol{\varepsilon}c^*_{jk} + \mathbf{h}d^*_{jk})\right]_{m\times p}\end{aligned}$$

From this definition we obtain

$$\begin{aligned}\widehat{A}\widehat{B} = {} & AA^* - BB^* + CB^* + BC^* + DD^* \\ & + (AB^* + BA^* + BD^* - DB^*)\mathbf{i} \\ & + (AC^* + CA^* + DC^* - CD^* + BD^* - DB^*)\boldsymbol{\varepsilon} \\ & + (AD^* + DA^* + CB^* - BC^*)\mathbf{h}\end{aligned}$$

Theorem 2.7. Let $\widehat{A} \in \mathbb{K}_n^m$, $\widehat{B}, \widehat{C} \in \mathbb{K}_r^n$, $\widehat{D} \in \mathbb{K}_t^r$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{K}$. The following equations hold:

i) $\widehat{A} \cdot (\widehat{B} \cdot \widehat{D}) = (\widehat{A} \cdot \widehat{B}) \cdot \widehat{D}$

ii) $\widehat{A} \cdot (\widehat{B} + \widehat{C}) = \widehat{A} \cdot \widehat{B} + \widehat{A} \cdot \widehat{C}$

iii) $(\widehat{B} + \widehat{C}) \cdot \widehat{D} = \widehat{B} \cdot \widehat{D} + \widehat{C} \cdot \widehat{D}$

iv) $\lambda(\widehat{A} \cdot \widehat{B}) = (\lambda\widehat{A}) \cdot \widehat{B}, (\widehat{A}\lambda) \cdot \widehat{B} = \widehat{A} \cdot (\lambda\widehat{B}), (\lambda_1\lambda_2)\widehat{A} = \lambda_1(\lambda_2\widehat{A})$

v) $\widehat{A} \cdot 0 = 0$

vi) $\widehat{A}\widehat{B} \neq \widehat{B}\widehat{A}$ (in general)

Proof. Straightforward. Q.E.D.

Corollary 2.8. $M_n(\mathbb{K})$ is module over $M_n(\mathbb{R})$. Moreover, for every hybrid matrix $\widehat{A} = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D \in M_n(\mathbb{K})$ can be written as follows:

$$\tilde{A} = \tilde{1}A + \tilde{\mathbf{i}}B + \tilde{\varepsilon}C + \tilde{\mathbf{h}}D \tag{2.2}$$

where $\tilde{1} = I_n, \tilde{\mathbf{i}} = iI_n, \tilde{\varepsilon} = \boldsymbol{\varepsilon}I_n, \tilde{\mathbf{h}} = \mathbf{h}I_n$. Additionally, it can be easily seen that

$$\tilde{\mathbf{i}}^2 = -\tilde{1}, \tilde{\varepsilon}^2 = \tilde{0}, \tilde{\mathbf{h}}^2 = \tilde{1}, \tilde{\mathbf{i}} \cdot \tilde{\mathbf{h}} = -\tilde{\mathbf{h}} \cdot \tilde{\mathbf{i}} = \tilde{\varepsilon} + \tilde{\mathbf{i}}.$$

So, we can write $M_n(\mathbb{K}) = sp\left\{\tilde{1}, \tilde{\mathbf{i}}, \tilde{\varepsilon}, \tilde{\mathbf{h}}\right\}$.

The set of hybrid matrices ($\mathbb{K}_n^m$) is a ring with matrix addition and ordinary matrix multiplication.

Definition 2.9. The trace of a square hybrid matrix $\widehat{A}$ is the sum of diagonal elements which is denoted by $tr(\widehat{A})$.

Theorem 2.10. The following conditions are satisfied:

i) For $\widehat{A} = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D \in \mathbb{K}_n^n$, $tr(\widehat{A}) = trA + \mathbf{i}trB + \boldsymbol{\varepsilon}trC + \mathbf{h}trD$

ii) For $\widehat{A}, \widehat{B} \in \mathbb{K}_n^n$, $tr(\widehat{A} + \widehat{B}) = tr(\widehat{A}) + tr(\widehat{B})$

iii) For any hybrid scalar $\tilde{k} \in \mathbb{K}, tr(k \cdot \widehat{A}) = k \cdot tr(\widehat{A})$

iv) For $\widehat{A}, \widehat{B} \in \mathbb{K}_n^n$, $tr(\widehat{A}\widehat{B}) = tr(\widehat{B}\widehat{A})$.

Proof. Here we will show the proof of i), ii), and iv).

i)
$$\begin{aligned} tr(\widehat{A}) &= \sum_{i=1}^{n}(A_{ii}) = \sum_{i=1}^{n}(a_{ii} + \mathbf{i}b_{ii} + \boldsymbol{\varepsilon}c_{ii} + \mathbf{h}d_{ii}) \\ &= trA + \mathbf{i}trB + \boldsymbol{\varepsilon}trC + \mathbf{h}trD \end{aligned}$$

ii)
$$\begin{aligned} tr(\widehat{A}) &= trA + \mathbf{i}trB + \boldsymbol{\varepsilon}trC + \mathbf{h}trD \\ tr(\widehat{B}) &= trA^* + \mathbf{i}trB^* + \boldsymbol{\varepsilon}trC^* + \mathbf{h}trD^* \\ tr\widehat{A} + tr\widehat{B} &= trA + trB + \mathbf{i}(trB + trB^*) \\ &\quad + \boldsymbol{\varepsilon}(trC + trC^*) + \mathbf{h}(trD + trD^*) \\ &= tr(A + A^*) + \mathbf{i}tr(B + B^*) + \boldsymbol{\varepsilon}tr(C + C^*) + \mathbf{h}tr(D + D^*) \\ &= tr(\widehat{A} + \widehat{B}) \end{aligned}$$

iv) It is known that $tr(AB) = tr(BA)$ for real matrices. With the help of this fact, it can be easily seen that $tr(\widehat{A} \cdot \widehat{B}) = tr(\widehat{B} \cdot \widehat{A})$.

Q.E.D.

Definition 2.11. The conjugate of hybrid matrix $\widehat{A} = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D$ is defined as $\overline{\widehat{A}} = A - \mathbf{i}B - \boldsymbol{\varepsilon}C - \mathbf{h}D$.

Theorem 2.12. Let $\widehat{A}$ and $\widehat{B}$ be hybrid matrices then the followings are satisfied for the property of conjugate:

i) $\overline{(\overline{\widehat{A}})} = \widehat{A}$

ii) $\overline{(\lambda\widehat{A})} = \overline{\lambda} \cdot \overline{\widehat{A}}, \lambda \in \mathbb{K}$

iii) $\overline{\widehat{A} + \widehat{B}} = \overline{\widehat{A}} + \overline{\widehat{B}}, \widehat{A}, \widehat{B} \in \mathbb{K}_n^m$

Definition 2.13. The transpose of $\widehat{A} = [z_{ij}] = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D \in \mathbb{K}_n^m$ is defined as

$$\widehat{A}^T = [z_{ij}]^T = [z_{ji}] \in \mathbb{K}_n^m \tag{2.3}$$

or

$$\widehat{A}^T = A^T + \mathbf{i}B^T + \boldsymbol{\varepsilon}C^T + \mathbf{h}D^T \in \mathbb{K}_n^m. \tag{2.4}$$

Theorem 2.14. For hybrid matrices $\widehat{A} \in \mathbb{K}_n^m$ and $\widehat{B} \in \mathbb{K}_p^n$, the transpose of hybrid matrices satisfies the followings:

i) $(\widehat{A}^T)^T = \widehat{A}$

ii) $(\lambda\widehat{A})^T = \lambda \cdot \widehat{A}^T, \lambda \in \mathbb{K}$

iii) $(\widehat{A} + \widehat{B})^T = \widehat{A}^T + \widehat{B}^T$

Proof. The proof of these properties can be seen by ordinary matrix operations. Q.E.D.

Theorem 2.15. For matrices $\widehat{A} \in \mathbb{K}_n^m$ and $\widehat{B} \in \mathbb{K}_p^n$,

$$(\widehat{A} \cdot \widehat{B})^T = \widehat{A}^T \cdot \widehat{B}^T. \tag{2.5}$$

Proof. It is known that $(A \cdot B)^T = B^T \cdot A^T$ for real matrices $A \in \mathbb{R}_n^m$ and $B \in \mathbb{R}_p^n$. By using this reality and the multiplication of two hybrid matrices, we can obtain the result which we desired. Q.E.D.

Definition 2.16. The conjugate transpose of hybrid matrix $\widehat{A} = [z_{ij}] = A + \mathbf{i}B + \boldsymbol{\varepsilon}C + \mathbf{h}D \in \mathbb{K}_n^m$ is defined as

$$\widehat{A}^* = [z_{ij}]^* = [\overline{z_{ji}}] \in \mathbb{K}_n^m \tag{2.6}$$

or in other way

$$\widehat{A}^* = A^T - \mathbf{i}B^T - \boldsymbol{\varepsilon}C^T - \mathbf{h}D^T \in \mathbb{K}_n^m. \tag{2.7}$$

Theorem 2.17. For proper hybrid matrices $\widehat{A}$ and $\widehat{B}$, the conjugate transpose operation satisfies the followings:

i) $\overline{(\widehat{A})}^T = \overline{(A^T)}$

ii) $(\lambda \widehat{A})^* = \overline{\lambda} \cdot \widehat{A}^*$

iii) $(\widehat{A} + \widehat{B})^* = \widehat{A}^* + \widehat{B}^*$

Proof. These equalities can be seen with using the properties of real matrices. Q.E.D.

3 Real matrix representation of hybrid matrices

In this section, we firstly define the real matrix representations of hybrid matrices. We then give some properties about that.
Let $\tilde{A} = A\tilde{1} + B\tilde{\mathbf{i}} + C\tilde{\varepsilon} + D\tilde{\mathbf{h}}$ be hybrid matrix. The linear map $\pounds_{\widehat{A}}$ can be defined as follows:

$$\begin{aligned} \pounds_{\widehat{A}}: \quad & \mathbb{K}_n^n \to \mathbb{K}_n^n \\ & \widehat{B} \to \pounds_{\widehat{A}}(\widehat{B}) = \widehat{A} \cdot \widehat{B} \end{aligned}$$

With the help of this mapping and the basis $\left\{\tilde{1}, \tilde{\mathbf{i}}, \tilde{\varepsilon}, \tilde{\mathbf{h}}\right\}$, we can write the followings:

$$\begin{aligned} \pounds_{\widehat{A}}(\tilde{1}) &= \widehat{A} \cdot \tilde{1} = A \cdot \tilde{1} + B \cdot \tilde{\mathbf{i}} + C \cdot \tilde{\varepsilon} + D \cdot \tilde{\mathbf{h}} \\ \pounds_{\widehat{A}}(\tilde{\mathbf{i}}) &= \widehat{A} \cdot \tilde{\mathbf{i}} = (C - B) \cdot \tilde{1} + (A - D) \cdot \tilde{\mathbf{i}} + C \cdot \tilde{\varepsilon} + D \cdot \tilde{\mathbf{h}} \\ \pounds_{\widehat{A}}(\tilde{\varepsilon}) &= \widehat{A} \cdot \tilde{\varepsilon} = B \cdot \tilde{1} + 0 \cdot \tilde{\mathbf{i}} + C \cdot \tilde{\varepsilon} + D \cdot \tilde{\mathbf{h}} \\ \pounds_{\widehat{A}}(\tilde{\mathbf{h}}) &= \widehat{A} \cdot \tilde{\mathbf{h}} = D \cdot \tilde{1} + B \cdot \tilde{\mathbf{i}} + (B - C) \cdot \tilde{\varepsilon} + A \cdot \tilde{\mathbf{h}} \end{aligned}$$

Then, we obtain following matrix:

$$\pounds_{\widehat{A}} = \begin{pmatrix} A & C-B & B & D \\ B & A-D & 0 & B \\ C & -D & A+D & B-C \\ D & C & -B & A \end{pmatrix}_{4n \times 4n} \tag{3.1}$$

Corollary 3.1. Matrix representation of $\tilde{1}, \tilde{\mathbf{i}}, \tilde{\varepsilon}$ and $\tilde{\mathbf{h}}$ can be found from (3.1) under the special cases as follows:

$$\pounds_{\tilde{1}} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}_{4n \times 4n}, \ \pounds_{\tilde{\mathbf{i}}} = \begin{pmatrix} 0 & -I_n & I_n & 0 \\ I_n & 0 & 0 & I_n \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{pmatrix}_{4n \times 4n}$$

$$\pounds_{\tilde{\varepsilon}} = \begin{pmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & -I_n \\ 0 & I_n & 0 & 0 \end{pmatrix}_{4n \times 4n}, \ \pounds_{\tilde{\mathbf{h}}} = \begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & -I_n & 0 & 0 \\ 0 & -I_n & I_n & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}_{4n \times 4n}$$

Moreover, it is easy to see that these real matrices satisfy the following relations:

$$\pounds_{\tilde{1}}^2 = I_{4n}, \pounds_{\tilde{\mathbf{i}}}^2 = -I_{4n}, \pounds_{\tilde{\varepsilon}}^2 = 0_{4n}, \pounds_{\tilde{\mathbf{h}}}^2 = I_{4n}$$

$$\pounds_{\tilde{\mathbf{i}}} \pounds_{\tilde{\mathbf{h}}} = -\pounds_{\tilde{\mathbf{h}}} \pounds_{\tilde{\mathbf{i}}} = \pounds_{\tilde{\varepsilon}} + \pounds_{\mathbf{i}}$$

Example 3.2. Let $\widehat{A} = \begin{bmatrix} 1+7\mathbf{i}+3\mathbf{h} & 2-\varepsilon \\ -5-3\mathbf{h} & \mathbf{i}+\varepsilon-\mathbf{i} \end{bmatrix}$ be a hybrid matrix. This matrix can be written as follows:

$$\widehat{A} = \begin{bmatrix} 1 & 2 \\ -5 & 0 \end{bmatrix} + \mathbf{i}\begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \mathbf{h}\begin{bmatrix} 3 & 2 \\ -3 & -1 \end{bmatrix}.$$

Then, the real matrix representation of hybrid matrix $\widehat{A}$ is

$$\pounds_{\widehat{A}} = \begin{bmatrix} 1 & 2 & -7 & -1 & 7 & 0 & 3 & 2 \\ -5 & 0 & 0 & 0 & 0 & 1 & -3 & -1 \\ 7 & 0 & -2 & 0 & 0 & 0 & 7 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & -2 & 4 & 4 & 7 & 1 \\ 0 & 1 & 3 & 1 & -8 & -1 & 0 & 0 \\ 3 & 2 & 0 & -1 & -7 & 0 & 1 & 2 \\ -3 & -1 & 0 & 1 & 0 & -1 & -5 & 0 \end{bmatrix}$$

Theorem 3.3. Let $\widehat{A}$ and $\widehat{B}$ be hybrid matrices $(\mathbb{K}_n^n)$. Then the following equations are satisfied:

i) $\pounds_{\tilde{I}_n} = \tilde{I}_{4n}$

ii) $\pounds_{\tilde{A}+\tilde{B}} = \pounds_{\tilde{A}} + \pounds_{\tilde{B}}$

iii) $\pounds_{\tilde{A}\cdot\tilde{B}} = \pounds_{\tilde{A}} \cdot \pounds_{\tilde{B}}$

iv) $\pounds_{\lambda\tilde{A}} = \lambda \cdot \tilde{A}$, $\lambda \in \mathbb{R}$

v) $\pounds_{\tilde{A}^T} \neq (\pounds_{\tilde{A}})^T$

Proof. The desired equations can be easily obtained with the help of above definitions and simple operations. Q.E.D.

4 Conclusion

In this study, we examine the hybrid matrices and their properties. This work is cover the other type of matrix systems such as complex matrices, dual matrices, and hyperbolic(perplex) matrices. That is why it can be regarded as a generalization of these matrix sets. In the last section, we present the real matrix representation of hybrid Matrices. Moreover, these representations can be extended to the right side multiplication.

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